

Laplace transforms:

Let $F: [0, \infty) \rightarrow \mathbb{R}$ be a f^n : Then Laplace transform of $F(t)$ is defined as

$$f(s) = \mathcal{L}[F(t)] = \int_0^{\infty} e^{-st} F(t) dt \quad t > 0$$

Q-1. Find $\mathcal{L}[F(t)]$ if $F(t) = c$

$$\begin{aligned} f(s) &= \int_0^{\infty} e^{-st} F(t) dt = \int_0^{\infty} e^{-st} \cdot c dt \\ &= c \int_0^{\infty} e^{-st} dt = c \left[\frac{e^{-st}}{-s} \right]_0^{\infty} \quad s > 0 \\ &= c \left[0 - \left(-\frac{1}{s}\right) \right] = \frac{c}{s} \end{aligned}$$

Q-2. Find $\mathcal{L}[F(t)]$ if $F(t) = t$

$$\begin{aligned} f(s) &= \int_0^{\infty} e^{-st} \cdot t dt = \left[t \frac{e^{-st}}{-s} \right]_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s} dt \\ &= -\frac{1}{s} \left[\frac{e^{-st}}{-s} \right]_0^{\infty} \\ &= \frac{1}{s} \left[0 - \left(-\frac{1}{s}\right) \right] = \frac{1}{s^2} \end{aligned}$$

Q-3. H.W. $\mathcal{L}[F(t)] = \frac{n!}{s^{n+1}}$ if $F(t) = t^n$

Q-4. $F(t) = e^{at}$

$$\begin{aligned} \mathcal{L}[F(t)] &= f(s) = \int_0^{\infty} e^{-st} e^{at} dt \\ &= \int_0^{\infty} e^{-(s-a)t} dt \quad \underline{s > a} \\ &= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} \end{aligned}$$

$$= \left[\frac{1}{-(s-a)} \right]_0^\infty$$

$$= \left[0 - \frac{1}{-(s-a)} \right] = \frac{1}{s-a}$$

Q. 5. $f(t) = \sin at$

$$f(s) = \int_0^\infty e^{-st} \sin at \, dt$$

$$f(s) = [f_1(s)]_0^\infty$$

$$\text{Let } f_1(s) = \int e^{-st} \sin at \, dt$$

$$= \sin at \frac{e^{-st}}{-s} - \int a \cos at \frac{e^{-st}}{-s} \, dt$$

$$= -\frac{1}{s} e^{-st} \sin at + \frac{a}{s} \int e^{-st} \cos at \, dt$$

$$= -\frac{1}{s} e^{-st} \sin at + \frac{a}{s} \left[\cos at \frac{e^{-st}}{-s} - \int -a \sin at \frac{e^{-st}}{-s} \, dt \right]$$

$$= -\frac{1}{s} e^{-st} \sin at - \frac{a}{s^2} \cos at e^{-st} - \frac{a^2}{s^2} \int e^{-st} \sin at \, dt$$

$$\left(1 + \frac{a^2}{s^2}\right) f_1(s) = -\frac{1}{s} e^{-st} \sin at - \frac{a}{s^2} e^{-st} \cos at \quad f(s)$$

$$\cancel{s^2 + a^2} f_1(s) = \frac{-\frac{1}{s} e^{-st} \sin at - a e^{-st} \cos at}{s^2}$$

$$f_1(s) = \frac{-e^{-st}}{s^2 + a^2} [s \sin at + a \cos at]$$

$$f(s) = [f_1(s)]_0^\infty = \left[\frac{-e^{-st}}{s^2 + a^2} (s \sin at + a \cos at) \right]_0^\infty$$

$$= -\frac{1}{s^2 + a^2} [0 - a] = \frac{a}{s^2 + a^2} \quad \text{Ans.}$$

$$= \frac{-1}{x^2 + y^2} \left[0 - a \right] = \frac{a}{x^2 + y^2} \quad \underline{\underline{\text{Ans.}}}$$

Orthogonal trajectory: Replace $\frac{dy}{dx} \rightarrow -\frac{dx}{dy}$

Eg: Find orthogonal trajectory of circles $x^2 + y^2 = a^2$

$$2x + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

$$\text{O.T.} \quad + \frac{dx}{dy} = +\frac{x}{y}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\log x = \log y + \log C$$

$$\log x = \log yC$$

$$x = yC \quad \underline{\underline{\text{Ans.}}}$$

THE END.